7.4.1

Sec 7.4 Integration of Rational Functions by Partial Fractions

Introduction & objectives

So far our techniques to evaluate an integral are as follows.

1. We know the answer; the integral is one of the known formulas.
2. Simplify. This involved simplifying a fraction or applying a Trig identity.
3. U-substitution
4. Integration by parts.

7.4.2

The idea of u-substitution was extended to include trigonometric integrals and trigonometric substitution. Now, we extend the idea of simplifying of an integral to integrals involving a rational func (ratio of two polynomials). We will simplify the rational function integrand using partial fractions and we will be able to integrate any integrate of the form

$\int \frac{P(x)}{D(x)}\,dx$ where $P(x)$ and $D(x)$ are polynomials. Of couse this method might not be the only method or shortest method of evaluation for a given integral.

7.4.3 → ← 7.4.4

Integration of Rational Functions by partial Fractions

1. Integration by reducing an improper fraction (long division)

$\frac{P(x)}{D(x)}$ where $\deg P(x) \geq \deg D(x)$

$$\begin{array}{r} Q(x) \\ D(x)\overline{)\;P(x)\;} \\ \vdots \\ \hline R(x) \end{array}$$

$\deg R(x) < \deg D(x)$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Ex Evaluate $\int \frac{-2x^2+x+2}{1-x}\,dx$

$$-x+1\overline{)\,-2x^2+x+2}$$

$$\int \frac{-2x^2+x+2}{1-x}\,dx = \int \left(2x+1+\frac{1}{1-x}\right)dx$$

7.4.5 $\rightarrow$ | $\leftarrow$ 7.4.6

2. Integration by partial fractions when the degree of numerator is less than the degree of the denominator.

2(i). The denominator can be factored into distinct linear factors

Ex Evaluate $\int \frac{1}{1-x^2}\,dx$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

There are two ways to find the constants in partial fractions.

(a). Multiply both sides by the LCD & set coeff of terms with the same degree equal to one another.

$$(1-x)(1+x)\left(\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}\right)$$

$$1 = A(1+x) + B(1-x)$$

$$1 = (A-B)x + (A+B)$$

$$\begin{cases} A-B=0 \\ A+B=1 \end{cases} \Rightarrow \cdots$$

7.4.7 → ← 7.4.8

(b) Multiply both sides by the LCD & plug in convenient values for the variable.

$$(1-x)(1+x)\left(\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}\right)$$

$$1 = A(1+x) + B(1-x)$$

For $x=-1$, $A(0) + B(2) = 1 \Rightarrow B = \frac{1}{2}$

For $x=1$, $A(2) + B(0) = 1 \Rightarrow A = \frac{1}{2}$

Although method (a) may be longer, in method (b) one may not realize if wrong partial fractions are used.

2(ii). The denominator can be factored into linear factors, where some of which are repeated.

Ex Evaluate $\int \frac{dx}{x^3+x^2}$.

$$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1} = \frac{Ax}{x^2} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

7.4.9 → 7.4.10

2(iii). The denominator containts dintinct irreducible quadratic factors.

Ex Evaluate $\int \frac{dx}{x^3+x}$.

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{\quad}{x} + \frac{\quad}{x^2+1}$$

2(iv). The denominator contains irreducible quadratic factors where some of which are repeated.

Ex Evaluate $\int \frac{x^2-3x+1}{(x^2+1)^2}\,dx$

$$\frac{x^2-3x+1}{(x^2+1)^2} = \frac{\quad}{x^2+1} + \frac{\quad}{(x^2+1)^2}$$

7.4.11 → ← 7.4.12

$$\frac{x^2-3x+1}{(x^2+1)^2} = \frac{1}{x^2+1} + \frac{-3x}{(x^2+1)^2}$$

CASE I ◦ The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the following example.

CASE II ◦ $Q(x)$ is a product of linear factors, some of which are repeated.
Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

CASE III ◦ $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\boxed{9} \qquad \frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

CASE IV ◦ $Q(x)$ contains a repeated irreducible quadratic factor.
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (9), the sum

$$\boxed{11} \qquad \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (11) can be integrated by first completing the square.

7.4.13→ ←7.4.14

Partial Fractions Method Applicable to other integrals

Ex Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$.

Let $u=\sqrt{x+4}$. Then $u^2=x+4$, $x=u^2-4$, and $dx=2u\,du$.

$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du$

7.4.15 → 7.4.16

In general, $\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C$.

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

More Examples

Ex Evaluate $\int \frac{3x^2-1}{x^3-x+5}\,dx$.

Let $u = x^3-x+5$. Then $du = (3x^2-1)\,dx$

$$\int \frac{3x^2-1}{x^3-x+5}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|x^3-x+5| + C$$

Ex Evaluate $\int \frac{x^4+x^2+1}{x^3+x}\,dx$

$$\begin{array}{r} x \\ x^3+0x^2+x \overline{)\, x^4+0x^3+x^2+0x+1} \\ \underline{x^4+0x^3+x^2} \\ 1 \end{array}$$

$$\frac{x^4+x^2+1}{x^3+x} = x + \frac{1}{x^3+x}$$

$$\int \frac{x^4+x^2+1}{x^3+x}\,dx = \int \left(x + \frac{1}{x^3+x}\right)dx = \int x\,dx + \int \frac{1}{x^3+x}\,dx$$

2nd integral was an earlier problem $= \frac{1}{2}x^2 + \ln|x| - \frac{1}{2}\ln(x^2+1) + C$

Ex Evaluate $\int \frac{x-1}{x^2(1+x^2)}\,dx$.

$$\frac{x-1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}$$

$$Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2 = x-1$$

7.4.17 →

$$Ax(1+x^2)+B(1+x^2)+(Cx+D)x^2=x-1$$

$$(A+C)x^3+(B+D)x^2+Ax+B=x-1$$

$$\begin{cases} A+C=0 \longrightarrow C=-A=-1 \\ B+D=0 \longrightarrow D=-B=+1 \\ A=1 \\ B=-1 \end{cases}$$

$$\frac{x-1}{x^2(1+x^2)}=\frac{1}{x}+\frac{-1}{x^2}+\frac{-x+1}{1+x^2}$$

$$\int\frac{x-1}{x^2(1+x^2)}dx=\int\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1-x}{1+x^2}\right)dx$$

$$=\int\frac{1}{x}dx-\int\frac{1}{x^2}dx+\int\left(\frac{1}{1+x^2}-\frac{x}{1+x^2}\right)dx$$

$$=\ln|x|-\int x^{-2}dx+\int\frac{1}{1+x^2}dx-\int\frac{x}{1+x^2}dx$$

← 7.4.18

Let $u=1+x^2$. Then $du=2x\,dx \Rightarrow x\,dx=\frac{du}{2}$

$$=\ln|x|-(-x^{-1})+\tan^{-1}x-\int\frac{1}{u}\frac{du}{2}$$

$$=\ln|x|+\frac{1}{x}+\tan^{-1}x-\frac{1}{2}\ln|u|+C$$

$$=\ln|x|+\frac{1}{x}+\tan^{-1}x-\frac{1}{2}\ln|1+x^2|+C$$

$$=\ln|x|+\frac{1}{x}+\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+C$$